The phase diagram for the Nambu–Jona-Lasinio model with 't Hooft and eight-quark interactions

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It is shown that the endpoint of the first order transition line which merges into a crossover regime in the phase diagram of the Nambu–Jona-Lasinio model, extended to include the six-quark 't Hooft and eight-quark interaction Lagrangians, is pushed towards vanishing chemical potential and higher temperatures with increasing strength of the OZI-violating eight-quark interactions. We clarify the connection between the location of the endpoint in the phase diagram and the mechanism of chiral symmetry breaking at the quark level. We show how the $8q$ interactions affect the number of effective quark degrees of freedom. We are able to obtain the correct asymptotics for this number at large temperatures by using the Pauli-Villars regularization.

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I. INTRODUCTION

The last two decades have witnessed great efforts towards the understanding of the QCD phase diagram, in terms of effective low energy theories paralleled by QCD lattice calculations, see e.g. the recent reviews [1]-[4], or the paper [5]. The domain of small to moderate baryonic chemical potential $0 < \mu < 400$ MeV and temperatures $0 < T < 200$ MeV, is of specific relevance for relativistic heavy ion collisions. Of fundamental importance in the study of the phase diagram are chiral symmetry and confinement, however the finite size $(m_u, m_d, m_s \neq 0)$ as well as the difference in the bare quark masses $(m_u \neq m_d \neq m_s)$ pose major problems both from the calculational point of view and in the implications due to deviations from the ideal situations, where the chiral condensate and the Polyakov loop are known to be the appropriate order parameters to characterize the phase state of the quark-gluon system.

The present study focuses on the chiral symmetry breaking aspects related to non-zero current quark mass values. Our arguments will be based on the successful model of Nambu–Jona-Lasinio (NJL) [6], combined with the $U_A(1)$ breaking $2N_f$ flavor determinant of 't Hooft [7]-[9] (NJLH). Moreover the most general $U(3)_L \otimes U(3)_R$ chiral invariant non derivative eight-quark ($8q$) interactions [10] are included. These terms were proven to render the effective potential of the NJLH model globally stable, and their effect has been thoroughly studied in low energy characteristics of pseudoscalar and scalar mesons [11], at finite temperature [12, 13] and in presence of a constant magnetic field [14]. These studies have lead at instances to sizeable and unforeseen effects. Of particular importance for the present work is that the strength of the $8q$ coupling is strongly correlated with the temperature and slope at which the crossover occurs and that it can be regulated together with the four-quark coupling, leaving the meson spectra at $T = 0$ unaffected (with exception of the scalar σ meson mass which decreases with increasing $8q$ coupling) [13]. As a result the symmetry breaking for large $8q$ couplings is induced by the $6q$ 't Hooft coupling strength, as opposed to the case with small $8q$ coupling, where the dynamical breaking of symmetry is controlled by the $4q$ coupling strength [11]. We would like to comment on a natural question which arises here, namely, can higher order many-quark interactions be also important? With regard to this, explicit argumets of A. A. Andrianov and V. A. Andrianov are known [15], which show that the structure of the QCD-motivated models at low energies with effective multi-fermion interactions and a finite cut-off in the chiral symmetry-breaking regime should contain only the vertices with four, six and eight-fermion interactions in four dimensions. This result explains partly the approximation used in our work.

Thus, in this paper we give a quantitative account of local multi-fermion forces on the phase diagram in the (T, μ) plane, by comparing the results for two sets of parameters, in the small and large coupling regimes of the $8q$ strengths, corresponding to the two above mentioned alternative mechanisms of chiral symmetry breaking.

Throughout the paper we work for simplicity in the isospin limit $m_u = m_d \neq m_s$, breaking explicitly the chiral $SU(3)_L \otimes SU(3)_R$ symmetry to the $SU(2)_I \otimes U(1)_Y$ (isospin-hypercharge) subgroup, and take the same baryonic chemical potential μ for all quark species. Generalizations to take into account the nonzero isospin chemical potential can be implemented as for instace in [16].

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II. EFFECTIVE LAGRANGIAN

The explicit form of the multi-quark Lagrangian considered is presented in [10, 11]

$$\mathcal{L}_{\rm eff} = \bar q(i\gamma^\mu\partial_\mu - m)q + \mathcal{L}_{4q} + \mathcal{L}_{6q} + \mathcal{L}^{(1)}_{8q} + \mathcal{L}^{(2)}_{8q}. \quad (1)$$

Quark fields q have color ($N_c = 3$) and flavor ($N_f = 3$) indices which are suppressed, $\mu = 0, 1, 2, 3$. Here

$$\mathcal{L}_{4q} = \frac{G}{2}\left[(\bar q\lambda_a q)^2 + (\bar q i\gamma_5\lambda_a q)^2\right], \quad (2)$$

$$\mathcal{L}_{6q} = \kappa(\det \bar q P_L q + \det \bar q P_R q), \quad (3)$$

$$\mathcal{L}^{(1)}_{8q} = 8g_1\left[(\bar q_i P_R q_m)(\bar q_m P_L q_i)\right]^2, \quad (4)$$

$$\mathcal{L}^{(2)}_{8q} = 16g_2\left[(\bar q_i P_R q_m)(\bar q_m P_L q_j)(\bar q_j P_R q_k)(\bar q_k P_L q_i)\right]. \quad (5)$$

The matrices acting in flavor space, $\lambda_a,\ a = 0, 1, \ldots, 8$, are normalized such that $\mathrm{tr}(\lambda_a\lambda_b) = 2\delta_{ab}$; $\lambda_0 = \sqrt{\frac{2}{3}}1$, and $\lambda_k,\ k = 1, 2, \ldots, 8$ are the standard $SU(3)$ Gell-Mann matrices; $P_{L,R} = (1 \mp \gamma_5)/2$ are chiral projectors and the determinant is over flavor indices. The large N_c behaviour of the model is reflected in the dimensionful coupling constants, $[G] = M^{-2}$, $[\kappa] = M^{-5}$, $[g_1] = [g_2] = M^{-8}$, which count as $G \sim 1/N_c$, $\kappa \sim 1/N_c^{N_f}$, and $g_1, g_2 \sim 1/N_c^4$ or less. As a result the NJL interaction (2) dominates over $\mathcal{L}_{6q}$ at large N_c, as one would expect, because Zweig's rule is exact at $N_c = \infty$. Let us note that the 8q-interaction $\mathcal{L}^{(1)}_{8q}$ breaks Zweig's rule as well.

Since the coupling constants G, κ, g_1, g_2 are dimensionful, the model is not renormalizable. We use the cut-off Λ to render quark loops finite. The global chiral $SU(3)_L \times SU(3)_R$ symmetry of the Lagrangian (1) at $m = 0$ is spontaneously broken to the $SU(3)$ group, showing the dynamical instability of the fully symmetric solutions of the theory. In addition, the current quark mass m, being a diagonal matrix in flavor space with elements $\mathrm{diag}(m_u, m_d, m_s)$, explicitly breaks this symmetry down, retaining only the reduced $SU(2)_I \times U(1)_Y$ symmetries of isospin and hypercharge conservation, if $m_u = m_d \neq m_s$.

The model has been bosonized in the framework of functional integrals in the stationary phase approximation leading to the following effective mesonic Lagrangian $\mathcal{L}_{\rm bos}$ at $T = \mu = 0$

$$\begin{aligned}
&\mathcal{L}_{\rm eff} \to \mathcal{L}_{\rm bos} = \mathcal{L}_{\rm st} + \mathcal{L}_{\rm ql}, \qquad \mathcal{L}_{\rm st} = h_a\sigma_a + \frac{1}{2}h^{(1)}_{ab}\sigma_a\sigma_b + \frac{1}{2}h^{(2)}_{ab}\phi_a\phi_b + \mathcal{O}(\text{field}^3),\\
&W_{\rm ql}(\sigma,\phi) = \frac{1}{2}\ln|\det D_E^\dagger D_E| = -\int\frac{d^4x_E}{32\pi^2}\sum_{i=0}^{\infty} I_{i-1}\mathrm{tr}(b_i) = \int d^4x_E\mathcal{L}_{\rm ql},\\
&b_0 = 1,\ \ b_1 = -Y,\ \ b_2 = \frac{Y^2}{2} + \frac{\Delta_{us}}{\sqrt{3}}\lambda_8 Y,\ \ \ldots,\\
&Y = i\gamma_\alpha(\partial_\alpha\sigma + i\gamma_5\partial_\alpha\phi) + \sigma^2 + \{\mathcal{M},\sigma\} + \phi^2 + i\gamma_5[\sigma + \mathcal{M},\phi]
\end{aligned} \quad (6)$$

written in terms of the scalar, $\sigma = \lambda_a\sigma_a$, and pseudoscalar, $\phi = \lambda_a\phi_a$, nonet valued quantum fields. The result of the stationary phase integration at leading order, $\mathcal{L}_{\rm st}$, is shown here as a series in growing powers of σ and ϕ. The result of the remaining Gaussian integration over the quark fields is given by $W_{\rm ql}$. Here the Laplacian in euclidean space-time $D_E^\dagger D_E = \mathcal{M}^2 - \partial_\alpha^2 + Y$ is associated with the euclidean Dirac operator $D_E = i\gamma_\alpha\partial_\alpha - \mathcal{M} - \sigma - i\gamma_5\phi$ (the γ_α, $\alpha = 1, 2, 3, 4$ are antihermitian and obey $\{\gamma_\alpha, \gamma_\beta\} = -2\delta_{\alpha\beta}$); $\mathcal{M} = \mathrm{diag}(M_u, M_d, M_s)$ is the constituent quark mass matrix (to explore the properties of the spontaneously broken theory, we define quantum fields σ_a, ϕ_a as having vanishing vacuum expectation values in the asymmetric phase).

The expression for the one-quark-loop action $W_{\rm ql}$ has been obtained by using a modified inverse mass expansion of the heat kernel associated to the given Laplacian [17]. The procedure takes into account the differences $\Delta_{us} = M_u^2 - M_s^2$ in the nonstrange and strange constituent quark masses in a chiral invariant way at each order of the expansion, b_i being the generalized Seeley–DeWitt coefficients of the new series. This modification distinguishes our calculation from the one made in [18]. In fact we consider the series up to and including the order b_2 that corresponds to the first nontrivial step in the expansion of the induced effective hadron Lagrangian at long distances. At this stage meson fields obtain their kinetic terms, but are still considered to be elementary objects. The information about their quark-antiquark origin enters only through the coefficients such as the average

$$I_i = \frac{1}{3}\left[J_i(M_u^2) + J_i(M_d^2) + J_i(M_s^2)\right] \quad (7)$$

over the 1-loop euclidean momentum integrals J_i with $i+1$ vertices ($i = 0, 1, \ldots$)

$$J_i(M^2) = 16\pi^2\Gamma(i+1)\int \frac{\mathrm{d}^4 p_E}{(2\pi)^4}\,\hat{\rho}_\Lambda \frac{1}{(p_E^2+M^2)^{i+1}}. \tag{8}$$

For the explicit evaluation of $J_i(M^2)$ we use the Pauli–Villars regularization method with two subtractions in the integrand. The procedure is fully defined by the insertion of the particular operator

$$\hat{\rho}_\Lambda = 1 - \left(1 - \Lambda^2 \frac{\partial}{\partial M^2}\right) \exp\left(\Lambda^2 \frac{\partial}{\partial M^2}\right). \tag{9}$$

Here the covariant cut-off Λ is a free dimensionful parameter which characterizes the scale of the chiral symmetry breaking in the effective model considered. To the order of the heat kernel series truncated, only the integrals J_0, J_1 are needed. These are quadratic and logarithmic divergent respectively with $\Lambda \to \infty$, all other J_i are finite. Note that the recurrence relation

$$J_{i+1}(M^2) = -\frac{\partial}{\partial M^2} J_i(M^2) \tag{10}$$

is fulfilled. If J_i is known for one value of i, then the function may be computed for other values of i by successive applications of the relation.

In $\mathcal{L}_{\rm st}$ the h_a are determined via the stationary phase conditions. These conditions and the pattern of explicit symmetry breaking show that in general h_a can have only three non-zero components at most with indices $a = 0, 3, 8$, i.e. $h_a\lambda_a = \mathrm{diag}(h_u, h_d, h_s)$, which can be found from a system of three independent equations

$$\begin{cases} Gh_u + \Delta_u + \dfrac{\kappa}{16}\, h_d h_s + \dfrac{g_1}{4}\, h_u(h_u^2+h_d^2+h_s^2) + \dfrac{g_2}{2}\, h_u^3 = 0, \\ Gh_d + \Delta_d + \dfrac{\kappa}{16}\, h_u h_s + \dfrac{g_1}{4}\, h_d(h_u^2+h_d^2+h_s^2) + \dfrac{g_2}{2}\, h_d^3 = 0, \\ Gh_s + \Delta_s + \dfrac{\kappa}{16}\, h_u h_d + \dfrac{g_1}{4}\, h_s(h_u^2+h_d^2+h_s^2) + \dfrac{g_2}{2}\, h_s^3 = 0. \end{cases} \tag{11}$$

Here $\Delta_f = M_f - m_f$, $f = u, d, s$. The matrix valued constants of higher order, like for instance $h_{ab}^{(1,2)}$ in $\mathcal{L}_{\rm st}$, are uniquely determined once the h_f are known [11, 19]. The stability of the effective potential is guaranteed if the system (11) has only one real solution. For that the couplings must fulfill the inequalities [10]: $g_1 > 0$, $g_1 + 3g_2 > 0$, $Gg_1 > (\kappa/16)^2$.

TABLE I. Parameters of the model: $m_u = m_d$, m_s (MeV), G (GeV^{-2}), Λ (MeV), κ (GeV^{-5}), g_1, g_2 (GeV^{-8}). We also show the corresponding values of constituent quark masses $M_u = M_d$ and M_s (MeV).

	m_u	m_s	M_u	M_s	Λ	G	$-\kappa$	g_1	$-g_2$
a	5.9	186	359	554	851	10.92	1001	1000*	47
b	5.9	186	359	554	851	7.03	1001	8000*	47

TABLE II. The masses, weak decay constants of light pseudoscalars (in MeV), the singlet-octet mixing angle θ_p (in degrees), and the quark condensates $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle, \langle\bar{s}s\rangle$ expressed as usual by positive combinations in MeV.

	m_π	m_K	m_η	$m_{\eta'}$	f_π	f_K	θ_p	$-\langle\bar{u}u\rangle^{\frac{1}{3}}$	$-\langle\bar{s}s\rangle^{\frac{1}{3}}$
a	138*	494*	477	958*	92*	117*	–14	235	187
b	138*	494*	477	958*	92*	117*	–14	235	187

TABLE III. The masses of the scalar nonet (in MeV), and the corresponding singlet-octet mixing angle θ_s (in degrees).

	$m_{a_0(980)}$	$m_{K_0^*(800)}$	$m_{f_0(600)}$	$m_{f_0(1370)}$	θ_s
a	980*	1201	691	1368	23
b	980*	1201	463	1350	19

In this paper we use the two parameter sets of Table I, which differ only in the choice of the $4q$ coupling G and the $8q$ strength g_1. Set (b) is the same as in [13] (there was a misprint in the value for the constituent strange quark mass, which we corrected). Tables II-III display the numerical fits at $T = \mu = 0$ (input is denoted by a *). The only difference in the observables of the two sets occurs in the singlet-octet flavor mixing channel of the scalars, mainly in the σ-meson (i.e. $f_0(600)$) mass. The model parameters are kept unchanged in the calculation of the T and μ dependent solutions of the gap equations (see next sections).

It is worthwhile to stress that there is an essential difference between the two alternative ground states chosen here as the configurations on top of which the $T \neq 0$ and $\mu \neq 0$ effects are studied: Case (a) corresponds to the standard picture of the NJL hadronic vacuum. In this picture chiral symmetry is spontaneously broken at $T = \mu = 0$ when $G > G_{crit}$. Case (b) corresponds to a new alternative, related to the pattern where $G < G_{crit}$. In this case chiral symmetry can be broken only due to the six-quark interactions, when $|\kappa|$ exceeds some critical value (the $8q$-interactions could in principle also induce symmetry breaking, however the mass spectra are then not well reproduced). One can hardly distinguish between the two cases at $T = \mu = 0$, the spectra of 0^{-+} and 0^{++} low-lying mesons do not show much difference: the model parameters are the same, except for the correlated G and g_1 values. The larger value of g_1 in the case (b) is a signal of the increasing role played by the eight-quark OZI-violating interactions, but this does not affect the value of the mixing angle θ_p, and only slightly diminishes θ_s. Such insensitivity follows from the observation that the stationary phase equations (11) and mass formulae of the light 0^{-+} and 0^{++} states [11] only depend on the couplings G and g_1 through the linear combination $\xi = G + g_1(h_u^2 + h_d^2 + h_s^2)/4$, except for the 00, 08 and 88 states inside the scalar nonet. However as soon as T or μ are finite, the h_f start to change due to their intrinsic T, μ dependence, acquired through the coupling to the quark loop integrals in the gap equations, eqs. (12) below. The T, μ dependence of the combination ξ above is steered by the strength g_1. This is the main reason why the $8q$-interactions may strongly affect the thermodynamic observables, without changing the spectra at $T = \mu = 0$.

III. THERMODYNAMIC POTENTIAL

A. Case of vanishing temperarure and chemical potential

Before addressing the thermodynamical potential it is instructive to briefly discuss the effective potential of the model at $T = \mu = 0$. Using standard techniques [19], we obtain from the gap-equations

$$h_f + \frac{N_c}{2\pi^2} M_f J_0(M_f^2) = 0 \tag{12}$$

the effective potential $V(M_f)$ as a function of three independent variables $M_f = \{M_u, M_d, M_s\}$. If the parameters of the model are fixed in such a way that eqs. (11) have only one real solution, the effective potential is

$$\begin{aligned} V(M_f) &= -\frac{1}{2}\int_0^{M_f} \sum_{f=u,d,s} h_f \mathrm{d}M_f - \frac{N_c}{4\pi^2}\int_0^{M_f} \sum_{f=u,d,s} M_f J_0(M_f^2)\mathrm{d}M_f \\ &= \frac{1}{16}\left(4Gh_f^2 + \kappa h_u h_d h_s + \frac{3g_1}{2}\left(h_f^2\right)^2 + 3g_2 h_f^4\right)\Bigg|_0^{M_f} + \frac{N_c}{8\pi^2}\sum_{f=u,d,s} J_{-1}(M_f^2), \end{aligned} \tag{13}$$

where $h_f^2 = h_u^2 + h_d^2 + h_s^2,\ h_f^4 = h_u^4 + h_d^4 + h_s^4$, and we extend definition (7) for index $i = -1$ with

$$J_{-1}(M^2) = -\int_0^{M^2} J_0(M^2)\mathrm{d}M^2 = -\frac{1}{2}\left(M^2 J_0(M^2) + \Lambda^4 \ln\left(1 + \frac{M^2}{\Lambda^2}\right)\right). \tag{14}$$

Here J_0 has the explicit form

$$J_0(M^2) = \Lambda^2 - M^2 \ln\left(1 + \frac{\Lambda^2}{M^2}\right) \tag{15}$$

for the given choice of regulator.

The first integral in (13) accounts for the leading order stationary phase contribution. The second integral describes the quark one-loop part. Since both integrands in (13) are exact differentials, the line integrals depend only on the end points. The low limit of the integrals is adjusted so that $V(0) = 0$ (to understand this, it is enough to notice that the power-series expansion of $V(M_f)$ at small M_f starts from $V(0)$; this term does not depend on M_f and, therefore, does not affect the physical content of the theory; so we simply subtract it, calculating the potential energy of the system with regard to the energy of the symmetric vacuum in the imaginary world of massless quarks).

B. Case of finite temperature and chemical potential

The extension to finite T and μ of the bosonized Lagrangian (6) is effected through the quark loop integrals J_i (see eq.(8)). Due to the recurrence relation (10) it is sufficient to get it just for one of them, J_0, by introducing the Matsubara frequencies, ω_n, and the chemical potential, μ, through the substitutions [20]

$$\int \mathrm{d}p_{0E} \to 2\pi T \sum_{n=-\infty}^{\infty}, \quad p_{0E} \to \omega_n - i\mu, \quad \omega_n = \pi T(2n+1). \tag{16}$$

Inserting (16) into J_0 we obtain

$$\begin{aligned} J_0(M^2) \to J_0(M^2,T,\mu) &= 16\pi^2 T \int \frac{\mathrm{d}^3\vec{p}_E}{(2\pi)^3}\hat{\rho}_\Lambda \sum_{n=-\infty}^{+\infty} \frac{1}{C_n - 2i\mu\pi T(2n+1)} \\ C_n &= E_p^2 + \pi^2 T^2 (2n+1)^2 - \mu^2, \quad E_p = \sqrt{M^2 + \vec{p}_E^{\,2}}. \end{aligned} \tag{17}$$

The sum over n is evaluated to give

$$\sum_{n=-\infty}^{+\infty} \frac{1}{C_n - i2\mu\omega_n} = \frac{1}{(2\pi T)^2} \sum_{n=-\infty}^{+\infty} \frac{1}{(n+a)(n+b)} = \frac{1}{4TE_p}\left(\tanh\frac{\mu+E_p}{2T} - \tanh\frac{\mu - E_p}{2T}\right) \tag{18}$$

where we use the abbreviations $a = \frac{1}{2} + \frac{i}{2\pi T}(E_p - \mu)$ and $b = \frac{1}{2} - \frac{i}{2\pi T}(E_p + \mu)$. Thus we have

$$J_0(M^2,T,\mu) = 4\int_0^\infty \mathrm{d}|\vec{p}_E||\vec{p}_E|^2\hat{\rho}_\Lambda \frac{1}{E_p}(1 - n_q - n_{\bar{q}}) = J_0(M^2) - 4\int_0^\infty \mathrm{d}|\vec{p}_E||\vec{p}_E|^2\hat{\rho}_\Lambda \frac{n_q + n_{\bar{q}}}{E_p} \tag{19}$$

where the quark, anti-quark occupation numbers are given by

$$n_q = \frac{1}{1 + e^{\frac{E_p - \mu}{T}}}, \quad n_{\bar{q}} = \frac{1}{1 + e^{\frac{E_p + \mu}{T}}}. \tag{20}$$

Notice that $J_0(M^2,0,0) = J_0(M^2)$, therefore the vacuum piece is well isolated from the matter part. The remaining integral containing the quark number occupation densities $n_q, n_{\bar{q}}$ is strictly finite, the Λ dependent terms being a remnant of the Pauli–Villars regularization scheme. At small T and $M \neq 0$ we have

$$\begin{aligned} \int_0^\infty \mathrm{d}|\vec{p}_E||\vec{p}_E|^2 \frac{n_q}{E_p} &\simeq \int_0^\infty \mathrm{d}|\vec{p}_E| \frac{|\vec{p}_E|^2}{E_p} e^{-\frac{E_p}{T}} = T\sqrt{2TM}e^{-\frac{M}{T}} \int_0^\infty \mathrm{d}x\sqrt{x}e^{-x}\sqrt{1 + \frac{xT}{2M}} \\ &= \sqrt{\frac{\pi M}{2}} T^{\frac{3}{2}} e^{-\frac{M}{T}} \left(1 + \frac{3T}{8M} + \dots\right). \end{aligned} \tag{21}$$

The special feature of this integral is that it vanishes exponentially with $T \to 0$.

Now we are ready to evaluate the thermodynamical potential. Indeed, the gap-equations at finite T and μ are

$$h_f + \frac{N_c}{2\pi^2} M_f J_0(M_f^2, T, \mu) = 0. \tag{22}$$

Consequently,

$$\begin{aligned} V(M_f,T,\mu) &= -\frac{1}{2}\int_0^{M_f} \sum_{f=u,d,s} h_f \mathrm{d}M_f - \frac{N_c}{4\pi^2}\int_0^{M_f} \sum_{f=u,d,s} M_f J_0(M_f^2,T,\mu)\mathrm{d}M_f + C(T,\mu) \\ &= \frac{1}{16}\left(4Gh_f^2 + \kappa h_u h_d h_s + \frac{3g_1}{2}\left(h_f^2\right)^2 + 3g_2 h_f^4\right)\Bigg|_0^{M_f} + \frac{N_c}{8\pi^2}\sum_{f=u,d,s} J_{-1}(M_f^2,T,\mu) + C(T,\mu), \end{aligned} \tag{23}$$

where the function $C(T,\mu)$ does not depend on M, and therefore cannot be determined from the gap equation; it will be found from other arguments in the end of this section, but obviously $C(0,0) = 0$. The integral $J_{-1}(M^2,T,\mu)$ is the immediate generalization of the $T = \mu = 0$ case (14)

$$J_{-1}(M^2,T,\mu) = -\int_0^{M^2} J_0(M^2,T,\mu)\mathrm{d}M^2 = J_{-1}(M^2) + J_{-1}^{\mathrm{med}}(M^2,T,\mu), \tag{24}$$

where the medium contribution to $J_{-1}(M^2,T,\mu)$ is

$$
\begin{aligned}
J^{\rm med}_{-1}(M^2,T,\mu) &= 4\int_0^{M^2}\mathrm{d}M^2\int_0^\infty \mathrm{d}|\vec{p}_E||\vec{p}_E|^2\hat{\rho}_\Lambda\frac{n_q+n_{\overline{q}}}{E_p} = 4\int_0^\infty \mathrm{d}|\vec{p}_E||\vec{p}_E|^2\hat{\rho}_{\Lambda\vec{p}_E}\int_0^{M^2}\mathrm{d}M^2\,\frac{n_q+n_{\overline{q}}}{E_p}\\
&= 8\int_0^\infty \mathrm{d}|\vec{p}_E||\vec{p}_E|^2\hat{\rho}_{\Lambda\vec{p}_E}\left(2\left(E_p(M)-E_p(0)\right)+T\ln\frac{n_{qM}n_{\overline{q}M}}{n_{q0}n_{\overline{q}0}}\right)\\
&= 8T\int_0^\infty \mathrm{d}|\vec{p}_E||\vec{p}_E|^2\hat{\rho}_{\Lambda\vec{p}_E}\ln\frac{\left(1+e^{-\frac{E_p(0)-\mu}{T}}\right)\left(1+e^{-\frac{E_p(0)+\mu}{T}}\right)}{\left(1+e^{-\frac{E_p(M)-\mu}{T}}\right)\left(1+e^{-\frac{E_p(M)+\mu}{T}}\right)}
\end{aligned}
\tag{25}
$$

with $n_{q0}, n_{\overline{q}0}$ and $n_{qM}, n_{\overline{q}M}$ referring to the occupation numbers for massless and massive particles correspondingly, $E_p(M)=\sqrt{M^2+\vec{p}_E^{\,2}}$, $E_p(0)=|\vec{p}_E|$. In spite of the fact that the integral $J^{\rm med}_{-1}(M^2,T,\mu)$ is convergent, we still keep the regularization $\hat{\rho}_\Lambda$ to be consistent. Note that the action of the operator $\hat{\rho}_\Lambda$ (see eq. (9)) on any smooth function, depending on M^2 through the energy, $f(E_p(M))$, can also be expressed in terms of momentum as $\hat{\rho}_\Lambda f(E_p)=\hat{\rho}_{\Lambda\vec{p}_E}f(E_p)$, where

$$
\hat{\rho}_{\Lambda\vec{p}_E} = 1-\left(1-\Lambda^2\frac{\partial}{\partial\vec{p}_E^{\,2}}\right)\exp\left(\Lambda^2\frac{\partial}{\partial\vec{p}_E^{\,2}}\right). \tag{26}
$$

Then, noting that, for instance,

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}|\vec{p}_E||\vec{p}_E|^2\hat{\rho}_{\Lambda\vec{p}_E}\ln\left(1+e^{-\frac{E_p(M)-\mu}{T}}\right)\\
&= \hat{\rho}_\Lambda\left(\frac{|\vec{p}_E|^3}{3}\ln\left(1+e^{-\frac{E_p(M)-\mu}{T}}\right)\Bigg|_0^\infty\right)-\hat{\rho}_\Lambda\int_0^\infty \mathrm{d}|\vec{p}_E|\frac{|\vec{p}_E|^3}{3}\frac{\partial}{\partial|\vec{p}_E|}\ln\left(1+e^{-\frac{E_p(M)-\mu}{T}}\right)\\
&= \hat{\rho}_\Lambda\int_0^\infty \mathrm{d}|\vec{p}_E|\frac{|\vec{p}_E|^4 n_{qM}}{3TE_p(M)} = \frac{1}{3T}\int_0^\infty \mathrm{d}|\vec{p}_E||\vec{p}_E|^4\hat{\rho}_{\Lambda\vec{p}_E}\frac{n_{qM}}{E_p(M)}
\end{aligned}
\tag{27}
$$

where we used the fact the surface term disappears, we get finally

$$
J^{\rm med}_{-1}(M^2,T,\mu) = -\frac{8}{3}\int_0^\infty \mathrm{d}|\vec{p}_E||\vec{p}_E|^4\hat{\rho}_{\Lambda\vec{p}_E}\left(\frac{n_{qM}+n_{\overline{q}M}}{E_p(M)}-\frac{n_{q0}+n_{\overline{q}0}}{E_p(0)}\right). \tag{28}
$$

It is important to realize that the expansion of the integral (28) for small values of T and at $\mu=0$ starts from the term

$$
J^{\rm med}_{-1}(M^2,T,0) = \frac{16}{3}\int_0^\infty\frac{|\vec{p}_E|^3\mathrm{d}|\vec{p}_E|}{1+e^{\frac{|\vec{p}_E|}{T}}}+\mathcal{O}(T^{\frac{5}{2}}M^{\frac{3}{2}}e^{-\frac{M}{T}}) = \frac{14}{45}\pi^4T^4+\mathcal{O}(T^{\frac{5}{2}}M^{\frac{3}{2}}e^{-\frac{M}{T}}). \tag{29}
$$

This leading contribution in T arises from the combination $\sim(n_{q0}+n_{\overline{q}0})$ and does not depend on the cut-off, i.e. the Pauli-Villars regulator $\hat{\rho}_{\Lambda\vec{p}_E}$ does not affect the leading order of the low-temperature asymptotics of the integral. To make this clear let us consider the typical integral in eq. (28)

$$
\int_0^\infty \mathrm{d}|\vec{p}_E||\vec{p}_E|^4\frac{n_{qM}}{E_p(M)} = \int_M^\infty\frac{|\vec{p}_E|^3\mathrm{d}E_p}{1+e^{\frac{E_p(M)}{T}}} = T^4\int_0^\infty \mathrm{d}x\frac{(x^2+2x\frac{M}{T})^{\frac{3}{2}}}{1+e^{x+\frac{M}{T}}}. \tag{30}
$$

If $M=0$, then the integral can be evaluated explicitly and is found to be $\frac{7\pi^4}{120}T^4$, leading us to the result (29). If $M\neq 0$, we obtain at once the estimate at small T

$$
\int_0^\infty \mathrm{d}x\frac{(x^2+2x\frac{M}{T})^{\frac{3}{2}}}{1+e^{x+\frac{M}{T}}} = \frac{3\sqrt{\pi}}{4}\left(2\frac{M}{T}\right)^{\frac{3}{2}}e^{-\frac{M}{T}}\left(1+\mathcal{O}(T)\right). \tag{31}
$$

We conclude that the integral vanishes exponentially for small T and does not contribute to the leading order term in eq. (29). It is then clear that the action of the Pauli-Villars regulator $\hat{\rho}_{\Lambda\vec{p}_E}$ on the integrand, which consists in subtracting the contribution of the massive Pauli-Villars states, will not affect the leading term as well.

One might worry that the low-temperature expansion of the integral (29) starts from the unphysical contribution which corresponds to the massless quark states. This fear would be valid if the potential had not the term $C(T,\mu)$.

Let us fix the M_f-independent function $C(T,\mu)$ in eq. (23) to avoid the problem. For that it is instructive to compare the matter quark-loop part of the thermodynamic potential obtained here (i.e. the $J_{-1}^{\rm med}$-part) with the corresponding result of the standard NJL approach. There is only one difference between such calculations: we use the Pauli-Villars subtractions instead of a 3-dimensional cut-off Λ_3. Therefore we can expect that if one removes the regularizations in both approaches ($\hat{\rho}_{\Lambda\vec{p}_E} \to 1$ and $\Lambda_3 \to \infty$) the finite matter part of the thermodynamic potentials must coincide. From this requirement of consistency we find

$$C(T,\mu) = -\frac{N_c}{\pi^2}\int_0^\infty \mathrm{d}|\vec{p}_E||\vec{p}_E|^4 \frac{n_{q0}+n_{\bar{q}0}}{E_p(0)} \;\to\; C(T,0) = -\frac{7N_cN_f}{180}\pi^2T^4. \tag{32}$$

As a result, the unwanted massless quark contribution to the vacuum energy at small T disappears.

C. Effective number of quark degrees of freedom

A quantity of interest related with the thermodynamic potential is the number of quark degrees of freedom present at a certain temperature. For that we consider the quark pressure difference from the zero-temperature value

$$\nu(T) = \frac{p(T)-p(0)}{\pi^2T^4/90}, \tag{33}$$

in order that the total $\nu(0)=0$. Here $p(T) = -V(M_f^*, T, \mu=0)$, where M_f^* denotes the gap equation solution at a given T. Dividing the pressure in eq. (33) by $\pi^2T^4/90$, the result is presented in Stefan-Boltzmann units, i.e. one explicitly counts the number of relevant degrees of freedom.

It is known that in the case of massless quarks of two and three flavors the fermionic degrees of freedom at $T > T_c$ are estimated as $\nu = (7/8)\times 3\times 2\times 4 = 21$ and $\nu = (7/8)\times 3\times 3\times 4 = 31.5$, respectively. The system studied here consists of three-types of light quarks: u, d, s. Hence ν is expected to be in the interval $21 < \nu < 31.5$ in the region $T > T_c$, where chiral symmetry is "restored" (up to the explicit symmetry breaking effects caused by the current quark masses). Indeed, the solid curves $\nu(T/T_c)$, plotted for the sets (a) and (b) in fig. 1, at some values $T/T_c > 1$ enter the interval and approach asymptotically the upper bound $\nu = 31.5$ at high T: in particular, we have already at $T/T_c = 2.5$ that $\nu(2.5) = 30.95$ (set (a)) and $\nu(2.5) = 30.25$ (set (b)). This is too fast as compared with the lattice estimates [21] in 2+1 flavor QCD, but the difference can be ascribed to the simplifications introduced by the model under consideration (the essential difference is that the NJL model does not possess the quark-confinement property of QCD).

We can gain some understanding of the asymptotic behavior of $\nu(T)$ by considering eqs. (19) and (24). Firstly, it is easy to see that the integral $J_0(M^2,T,\mu)\to 0$ at $T\to\infty$. The reason for this is very simple and is contained in the integrand $(1-n_q-n_{\bar{q}})$ which vanishes at $T\to\infty$, as it explicitly follows from eqs. (20). Secondly, eq. (24) contains J_0 as an integrand. Thus we conclude that $J_{-1}(M^2,T,\mu)\to 0$ at $T\to\infty$. Next, from the gap equation (22) it follows that $h_f(T)\to 0$ at $T\to\infty$. Therefore $V(M_f,T,\mu)\sim C(T,\mu)$ at large T, i.e. we can say that the asymptotics is totally determined by the term $C(T,\mu)$, yielding $\nu(T\to\infty) = 31.5$, which is independent of any model parameters. This conclusion is attractive because it agrees with the general arguments of the previous paragraph. In fact, to many readers our conclusion that $\nu(T)$ has the correct asymptotics, may seem to be a much more compelling argument for fixing $C(T,\mu)$ than the assumption made that the different cut-off procedures must give the same result when cut-offs are removed.

A clear insight into the origin of this result can be obtained by considering the contribution of $C(T,\mu)$ and the term $\sim (n_{q0}+n_{\bar{q}0})$ in eq. (28) to the number of effective degrees of freedom $\nu(T)$, because exactly these terms determine the correct low-temperature behavior of the function $\nu(T)$. We designate this contribution by $\nu_\Lambda(T)$, and plot it in fig. 2. Thus, we consider the following function of temperature

$$\nu_\Lambda(T) = \frac{90N_c}{\pi^4T^4}\int_0^\infty |\vec{p}_E|^4\mathrm{d}|\vec{p}_E|\,(1-\hat{\rho}_{\Lambda\vec{p}_E})\frac{n_{q0}+n_{\bar{q}0}}{E_p(0)} \tag{34}$$

calculated at a fixed value of the cut-off parameter Λ (the value taken is the same for both parameter sets considered, see Table I). At first glance one might wish to associate $\nu_\Lambda(T)$ with the contribution of massless states, introduced to $J_{-1}^{\rm med}$ by assigning the low limit $M=0$ to the integral (24). However, this expectation is potentially fallacious. One can easily see from eq. (34) that $\nu_\Lambda(T)$ vanishes for all T if the the integral is not regularized $\hat{\rho}_{\Lambda\vec{p}_E}\to 1$. Hence only the auxiliary Pauli-Villars states of mass Λ contribute to $\nu_\Lambda(T)$

$$\nu_\Lambda(T) = \frac{90N_c}{\pi^4T^4}\int_0^\infty |\vec{p}_E|^4\mathrm{d}|\vec{p}_E|\left(1-\Lambda^2\frac{\partial}{\partial\vec{p}_E^{\,2}}\right)\frac{n_{q\Lambda}+n_{\bar{q}\Lambda}}{E_p(\Lambda)}. \tag{35}$$

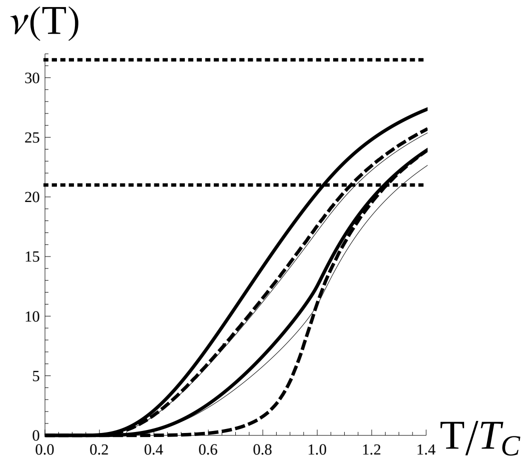

FIG. 1: The number of effective degrees of freedom associated with $N_f = 3$ flavor quarks as a function of T/T_c at $\mu = 0$ in the NJL model with 't Hooft and eight-quark interactions (bold solid curves correspond to our calculations with a finite cut-off, and thin solid curves show the same patterns when one removes the regulator in the thermal quark energy integral) compared with the result of Fukushima [23] (long dashes). The critical temperature is $T_c = 190\,\mathrm{MeV}$ (finite cut-off), $T_c = 179\,\mathrm{MeV}$ (no regulator) for parameter set (a) (upper bold solid line, and upper thin solid line, correspondingly), and $T_c = 135\,\mathrm{MeV}$ (finite cut-off), $T_c = 132\,\mathrm{MeV}$ (no regulator)for set (b) (lower bold solid line, and lower fine solid line, correspondingly). The long-dashed-curves are taken from fig. 3 of paper [23]: the upper one corresponds to the NJL model with 't Hooft interactions; the lower one to the NJL model with the Polyakov loop. The short-dashed horizontal lines correspond to the asymptotic high-temperature Stefan-Boltzmann (ideal gas) limit for the theory of massless fermions with three and two flavors respectively.

This contribution is given by an integral with a positive integrand. Here we have two dimensionful parameters, T and Λ. Therefore, at large T the series for $\nu_\Lambda(T)$ can be organized in powers of the dimensionless ratio Λ/T, i.e.

$$\nu_\Lambda(T) = \frac{180N_c}{\pi^4}\int_{\frac{\Lambda}{T}}^{\infty} \mathrm{d}x \frac{(x^2 - \frac{\Lambda^2}{T^2})^{\frac{3}{2}}}{1+e^x}\left[1 + \frac{\Lambda^2}{2T^2x^2}\left(1 + \frac{xe^x}{1+e^x}\right)\right] = \frac{21N_c}{2}\left[1 + \mathcal{O}\left(\frac{\Lambda}{T}\right)\right]. \tag{36}$$

A useful observation is that although the heavy mass states determine the value of the integral, it still has the correct asymptotics at large T, which does not depend on Λ. In other words, the integral $\sim (n_{q0} + n_{\bar{q}0})$ in eq. (28) must vanish at $T \to \infty$, in order that the function $\nu(T)$ has the right asymptotic behavior. This really happens, due to the Pauli-Villars regulator $\hat{\rho}_{\Lambda\vec{p}_E}$. In fig. 2 one can see that up to around $T \sim 100\,\mathrm{MeV}$ the contribution to the effective number of degrees of freedom is practically constant and nearly zero, then it increases and reaches asymptotically the value 31.5. The constant dashed curve shows the same quantity if the integral is not regularized. Such a big difference in the behavior of $\nu_\Lambda(T)$ translates in $\nu(T)$ (see fig. 1) to an enhancement in the number of quark degrees of freedom around the critical temperature in the presence of a regulator.

The conclusion that the Pauli-Villars regularization leads to the correct asymptotics is attractive because in the NJL model with 3-dimensional cut-off the Stefan-Boltzmann limit can be reached only when the cut-off is removed in the matter integrals [22], which are finite in themselves. Such a selective removal of the cut-off must be taken "cum grano salis", since technically the NJL model requires the presence of a finite ultraviolet cut-off throughout all integrals. Nevertheless, one might think of criticizing our result on the grounds that we seem to get the correct asymptotics due to the auxiliary and therefore unphysical Pauli-Villars terms. This is certainly not true. To see this let us return to eq. (28) and consider now the contribution of the term $\sim (n_{qM} + n_{\bar{q}M})$ to $\nu(T)$. (Our arguments will be based on considering the simplified case $M_u^* = M_d^* = M_s^* = M_*$. We need not be rigorous in the following discussion, because the unitary symmetry breaking effects are unimportant for the asymptotics.) The integral describes the thermal energy of massive quarks, but does not contribute at large T

$$\nu_{M_*}(T) = \frac{90N_c}{\pi^4T^4}\int_0^\infty |\vec{p}_E|^4 \mathrm{d}|\vec{p}_E|\hat{\rho}_{\Lambda\vec{p}_E}\frac{n_{qM_*} + n_{\bar{q}M_*}}{E_p(M_*)} = \frac{21N_c}{2}\left[1 - 1 + \mathcal{O}\left(\frac{M_*}{T}, \frac{\sqrt{\Lambda^2 + M_*^2}}{T}\right)\right]. \tag{37}$$

The vanishing result is a consequence of a total cancellation of two contributions: the first, 1, in the square brackets represents the contribution of the physical states with mass M_*; the second, -1, comes from the Pauli-Villars regulator. What is interesting here is that the second term, if one joins it with the other unphysical contributions of eq. (36),

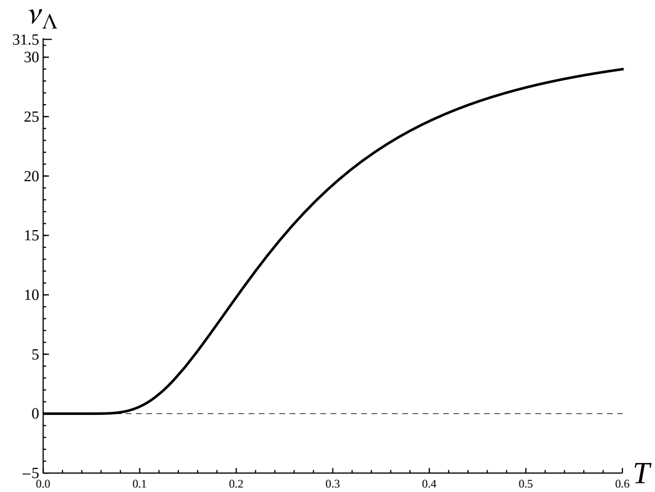

FIG. 2: Solid line: degrees of freedom due to the Pauli-Villars states, $\nu_\Lambda(T)$, at zero chemical potential, as function of T [GeV]. Dashed line: the same quantity at $\Lambda \to \infty$.

cancels them entirely. In other words, the correct large T asymptotics in (33) can equally be assigned to the pure physical states contribution as well.

Let us compare our results with the recent estimates of Fukushima [23], made on the basis of the three-flavor NJL model with and without the Polyakov loop. The starting values are very similar: our curve for the set (a) agrees well with the Fukushima estimate made in the standard NJL model approach with the 't Hooft six-quark interactions. Indeed, if we remove the cut-off dependence in the quark number occupation integrals (28), like it has been done in [23], the curves almost coincide: compare the upper thin solid line of our set (a), where the regulator has been removed, with the long dashed curve of Fukushima, almost on top of it in fig. 1. Taking systematically into account the finite value of the cut-off, we obtain the upper bold curve corresponding to the set (a). The set (b), compared with set (a), shows a rather strong (more than 50%) suppression of the abundant artificial quark excitations at $T/T_c > 0.2$ due to the large OZI-violating eight-quark interactions (see lower bold curve in fig. 1 for the finite cut-off result and lower thin solid line, where the cut-off condition has been relaxed. Notice that in both cases (a) and (b) the finite cut-off leads to larger values of ν, because $\nu_\Lambda(T)$ is non-zero and positive at $\Lambda = \infty$; this effect is more pronounced in the case of set (a)). Although with set (b) the model still fails to describe accurately the pressure at $0.3 < T/T_c < 1$, it leads to an improved description of the number of quark degrees of freedom as compared to set (a). Thus the OZI-violating interactions could potentially play an important role in the description of quark excitations. At least one should not exclude the set (b) if one includes the Polyakov loop effects in the framework of effective NJL-type models with multi-quark interactions, see [24] for results with the 3-dimensional cut-off.

IV. PHASE DIAGRAM

Fig. 3 shows the phase diagrams for the two sets of parameters. One observes a larger window for the first order transition regime in the case of stronger $8q$ interaction coupling g_1, the critical endpoint is situated at $(\mu_E, T_E) = (155, 108)$ MeV, whereas for the smaller coupling it is at $(\mu_E, T_E) = (338, 53)$ MeV. The temperature at zero chemical potential, in the crossover regime, is substantially smaller for the large $8q$ coupling case, around $T_c \simeq 135\,\mathrm{MeV}$ compared to $T_c \simeq 193\,\mathrm{MeV}$ for the sets (b) and (a) respectively [12]. The approximate values for T_c were obtained as usual, through the condition $\mathrm{d}^2M/\mathrm{d}T^2 = 0$ of strongest change in the slope of the constituent quark masses.

The length of the line of first order transitions can be increased further with increase of the $8q$ coupling g_1 and after some critical value it goes all the way through to the $\mu = 0$ end. A corresponding set of parameters has been considered in [13]. In this way we obtain the thin solid line in figs. 3, 4 that connects all CEP obtained by varying g_1 in the allowed interval for stability of the effective potential.

However we point out that also in the case of strong $8q$ couplings our results have with other approaches in common that the dynamical u and d quark masses suffer a considerable reduction at the transition temperatures, while the strange quark mass remains roughly the same. For instance for set (b) at $\mu = 170\,\mathrm{MeV}$ at the first order transition temperature $T = 103\,\mathrm{MeV}$, M_u jumps from $M_u = 208\,\mathrm{MeV}$ to $128\,\mathrm{MeV}$, while the strange quark mass changes only

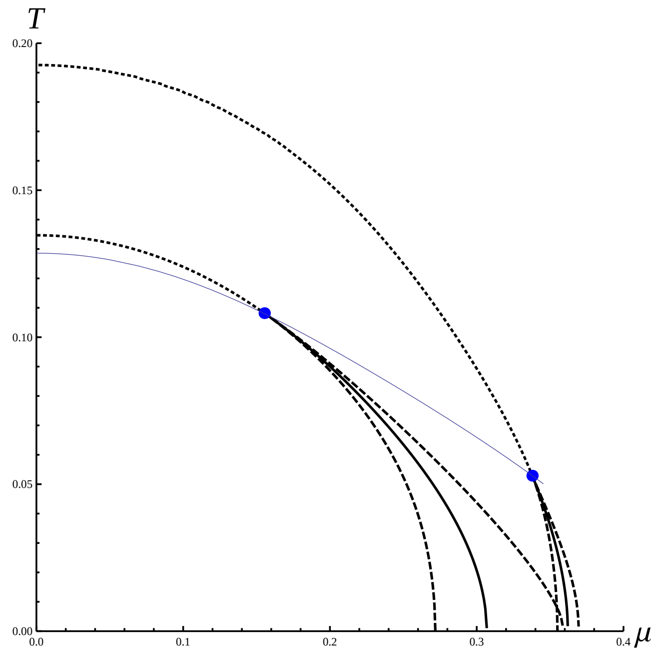

FIG. 3: Phase diagrams indicating first order phase transition lines: the short solid line corresponds to parameter set (a) of weak $8q$ coupling constant; the long solid line is for set (b) with strong $8q$ coupling. Dashed lines are spinodals. Circles (blue online) indicate the critical endpoints (CEP) for sets (a) and (b) respectively and dotted lines correspond to the crossover region. The thin solid line (blue online) represent all CEP obtained by varying the $8q$ coupling g_1 and keeping as before the meson mass spectra at $T = \mu = 0$ unchanged, except for the σ-meson mass. All units are in GeV.

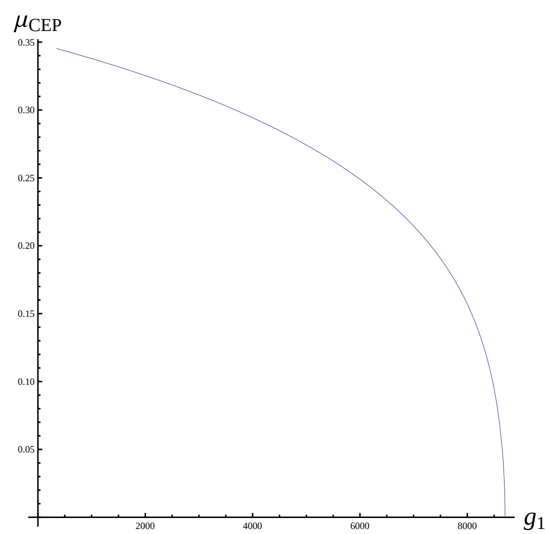

FIG. 4: The line of CEP described in fig. 3, now in the μ_{CEP}, g_1 plane.

from $M_s = 487\,\text{MeV}$ to $448\,\text{MeV}$. By increasing the temperature keeping μ fixed, one obtains at $T = 160\,\text{MeV}$ the values $M_u = 24\,\text{MeV}$ and $M_s = 381\,\text{MeV}$. Profiles of the quark masses are given in [25] for the $T = 0, \mu \neq 0$ case. The slow decrease of the strange quark mass is present in studies involving realistic values for the strange current quark mass [4].

The position of the critical endpoint in the three flavor NJL model with $U_A(1)$ breaking has been analyzed in [23], with the parameter set of [26], $(\mu_E, T_E) = (324, 48)\,\text{MeV}$ in comparison with the case with inclusion of the Polyakov loop, $(\mu_E, T_E) = (313, 102)\,\text{MeV}$. The increase of T_E is explained by the suppression of the artificial quark excitations at finite temperature and density in the presence of the Polyakov loop contribution [23]. Our set (a) with $(\mu_E, T_E) = (338, 53)\,\text{MeV}$ is in reasonable agreement with the above estimates.

Our result for set (b), $(\mu_E, T_E) = (155, 108)\,\mathrm{MeV}$, yields a twice lower value for the critical μ_E as compared to set (a). This new feature is related with the OZI-violating eight-quark interactions: it is well-known that NJL-type models without $8q$-forces have the tendency in common to lead to a critical endpoint at relatively high density, above $\mu_E \sim 300\,\mathrm{MeV}$ [23]. The small value found here is a good indicator that the main force responsible for dynamical chiral symmetry breaking has changed, being now associated with the 't Hooft $6q$-interactions.

Let us also take notice of the difference between the critical temperatures for the case (b), $T_c = 108\,\mathrm{MeV}$, and the Fukushima's case with Polyakov loop, $T_c = 204.8\,\mathrm{MeV}$. We expect that the inclusion of the Polyakov loop in our analysis will increase somewhat the value of T_c.

The NJL model with $8q$ interactions and without $U_A(1)$ symmetry breaking has been analyzed as well for the $SU(2)$ flavor case [27, 28]. Although the $8q$ forces are not needed to stabilize the effective potential of the model in the two flavor case, the same tendency as above in i) was observed, for instance: $(\mu_E, T_E) = (276, 62)\,\mathrm{MeV}$ with the $8q$ interactions, whereas $(\mu_E, T_E) = (330, 47)\,\mathrm{MeV}$ without them [27].

Another variant without $8q$ terms of the two flavor NJL model, including the vector-isoscalar interactions, which induce an effective chemical potential, has been considered a long time ago [29]. In this case, the critical endpoint is located at larger μ and lower temperatures $(\mu_E, T_E) = (350, 40)\,\mathrm{MeV}$. This effect has been studied also recently in [23, 27].

Further results concerning the position of the critical endpoint within other model approaches is given in the review [3] (note the notation there is in terms of $\mu_B = 3\mu$), in comparison with lattice results and the freezeout points extracted from heavy-ion experiments. For our two sets the critical endpoints are situated in case (a) slightly above and in case (b) slightly below the freezeout points obtained at different collision energies [3, 30].

V. CONCLUSIONS

The thermodynamic potential of the three flavor NJL model with 't Hooft and $8q$ interactions has been obtained in stationary phase approximation (at leading order) and using the Pauli-Villars regularization in quark loops (at one-loop level).

The main conclusions of this work can be classified as follows.

Firstly, we argued that a non-renormalizable effective theory at finite temperature and chemical potential, can be self-consistently studied with the use of the Pauli-Villars regularization. The importance of the regulator in the matter parts for consistency has also been discussed in [32] in connection with correlators. Instead, our present study is devoted to the construction of the thermodinamic potential. Unlike the conventional approach with 3-dimensional cut-off, the Pauli-Villars technique leads to the right asymptotic behavior of relevant thermodynamic observables. It is one of the main results of this paper.

Secondly, we quantified the effect of the new $8q$ terms on the number of degrees of freedom and on the phase diagram. We observe that in the large $8q$ coupling regime a strong depletion of the number of degrees of freedom below T_c is reached in comparison with the weak coupling case, working in the same direction as the effect produced through inclusion of the Polyakov loop in the model without $8q$ interactions.

Thirdly, we conclude that the NJL model with the $8q$ stabilizing interactions does not impede the possibility of having a phase diagram consisting only of first order transitions even for realistic quark masses. This will depend on the strength of the OZI-violating $8q$ interactions. At $\mu = 0$ there is growing evidence from lattice calculations that the transition is a crossover [31]. This would set an upper limit for the $8q$ coupling, which nevertheless can be sufficiently strong to trigger the first order transitions regime at low values of $\mu \neq 0$. This point deserves to be studied more carefully especially because it can help us to clarify the quark dynamics which is responsible for the mechanism of spontaneous chiral symmetry breaking. Indeed as it is shown above the shift of the critical endpoint to lower values of μ_E is possible only when the $6q$ interactions are responsible for the chiral phase transition (set (b)); however if the $4q$ coupling G exceeds its critical value, the $4q$ interactions drive the chiral phase transition in NJL-type models and as a result the critical endpoint is located at large values of μ_E (set (a)).

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